Math 247A Lecture 18 Notes

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1 The Mikhlin Multiplier Theorem

1.1 The Hilbert transform

Recall the **Hilbert transform**

$$Hf(x) = PV \int \frac{f(x-y)}{\pi y} \, dy.$$

We claimed that $\widehat{Hf}(\xi) = -i \operatorname{sgn}(\xi) \widehat{f}(\xi)$. Let

$$\widehat{f}_a(\xi) = \begin{cases} e^{-a\xi} & \xi > 0 \\ 0 & \xi \leq 0, \end{cases} \qquad \widehat{g}_a(\xi) = \begin{cases} 0 & \xi > 0 \\ e^{a\xi} & \xi \leq 0 \end{cases}$$

Then

$$(\widehat{f}_a - \widehat{g}_a)(\xi) = \begin{cases} e^{-a\xi} & \xi > 0 \\ 0 & \xi = 0 \\ -e^{a\xi} & \xi < 0 \end{cases} \xrightarrow{\mathcal{S}'(\mathbb{R}),\, a \to 0} \begin{cases} 1 & \xi > 0 \\ 0 & \xi = 0 \\ -1 & \xi < 0 \end{cases} = \operatorname{sgn}(\xi).$$

So we get

$$f_a(x) - g_a(x) \xrightarrow{\mathcal{S}'(\mathbb{R}),\, a \to 0} \operatorname{sgn}^\vee.$$

Now compute

$$f_a(x) = \int_0^\infty e^{2\pi i x \xi} e^{-a\xi} \, d\xi = \frac{1}{a - 2\pi i x},$$

$$g_a(x) = \int_{-\infty}^0 e^{2\pi i x \xi} e^{a\xi} \, d\xi = \frac{1}{a + 2\pi i x},$$

so

$$f_a(x) - g_a(x) = \frac{4\pi i x}{a^2 + 4\pi^2 x^2}.$$

Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$, and compute

$$\lim_{a\to 0}(f_a - g_a)(\varphi) = \lim_{a\to 0}\int \frac{4\pi i x}{a^2+4\pi^2x^2}\varphi(x)\,dx$$

We can't pull in the limit as is. We need the integrand to vanish near 0.

$$\begin{aligned}
&= \lim_{a\to 0}\int \frac{4\pi i x}{a^2+4\pi^2x^2}[\varphi(x)-\varphi(0)\mathbb{1}_{[-\varepsilon,\varepsilon]}(x)]\,dx \\
&= \lim_{a\to 0}\int_{-\varepsilon}^{\varepsilon} \frac{4\pi i x}{a^2+4\pi^2x^2}[\varphi(x)-\varphi(0)] + \lim_{a\to 0}\int_{|x|>\varepsilon}\frac{4\pi i x}{a^2+4\pi^2x^2}\varphi(x)\,dx \\
&= \int_{-\varepsilon}^{\varepsilon}\frac{i}{\pi x}[\varphi(x)-\varphi(0)] + \underbrace{\int_{|x|>\varepsilon}\frac{i}{\pi x}\varphi(x)\,dx}_{\xrightarrow{\varepsilon\to 0} i\,\mathrm{PV}(\frac{1}{\pi x})(\varphi)}.
\end{aligned}$$

On the other hand,

$$\left|\int_{-\varepsilon}^{\varepsilon}\frac{i}{\pi x}(\varphi(x)-\varphi(0))\,dx\right| \lesssim \varepsilon\|\varphi\|_{L^\infty} \xrightarrow{\varepsilon\to 0} 0.$$

So

$$f_a - g_a \xrightarrow{\mathcal{S}'(\mathbb{R}),a\to 0} i\,\mathrm{PV}\left(\frac{1}{\pi x}\right)(\varphi) = iH,$$

where $i\widehat{H} = \mathrm{sgn}$.

1.2 Littlewood-Paley projections and the Mikhlin multiplier theorem

Let's construct a dyadic partition of unity. Let $\varphi : \mathbb{R}^d \to [0,1]$, $\varphi \in C_c^\infty$ with

$$\varphi(x) = \begin{cases} 1 & |x| \le 1.4 \\ 0 & |x| > 1.42. \end{cases}$$

Let $\psi(x) = \varphi(x) - \varphi(2x)$; if we graph ψ, it is 0 before 0.7, increases quickly to 1 between 0.7 and 0.71, plateaus on 0.71 to 1.4, and goes down to 0 by 1.42.

For $N \in 2^{\mathbb{Z}}$, let $\psi_N 9x) = \psi(x/N)$. Note that

$$\sum_{N\in 2^{\mathbb{Z}}}\psi_N(x) = 1$$

a.e. (in fact for all $x \neq 0$.

Definition 1.1. The **Littlewood-Paley projection** to frequencies $|\xi| \sim N$ is given by

$$\widehat{P_N f}(\xi) = \widehat{f}(\xi)\psi_N(\xi), \qquad \text{i.e. } P_N f = f * [N^d \psi^\vee(N\,\cdot)].$$

We also define

$$\widehat{P_{\leq N} f}(\xi) = \widehat{f}(\xi)\varphi(\xi/n), \qquad \text{i.e. } P_{\leq N} f = [N^d \varphi^\vee(N\,\cdot)] * f$$

Remark 1.1. Caution: P_N is not a true projection since $P_N^2 = P_N$.

We can also define

$$P_{>n} = \operatorname{Id} - P_{\leq N}, \qquad P_{M\leq\cdot\leq N} = \sum_{M\leq K\leq N} P_K.$$

Theorem 1.1 (Mikhlin multiplier theorem)**.** *Let $m : \mathbb{R}^d \backslash \{0\} \to \mathbb{C}$ be such that $|D_\xi^\alpha m(\xi)| \lesssim |\xi|^{-|\alpha|}$ uniformly for $|\xi| \neq 0$ and $0 \leq |\alpha| \leq \lceil \frac{d+1}{2} \rceil$. Then*

$$f \mapsto [m(\xi)\widehat{f}(\xi)]^\vee = m^\vee * f$$

is bounded on L^p for all $1 < p < \infty$.

Proof. Taking $\alpha = 0$, we get $M \in L^\infty$. By Plancherel,

$$\|m^\vee * f\|_2 = \|m\widehat{f}\|_2 \leq \|m\|_{L^\infty}\|\widehat{f}\|_2 \lesssim \|f\|_2.$$

It suffices to check the regularity condition (c) is satisfied by the kernel $m^\vee$. We'll first prove this assuming $|D_\xi^\alpha m(\xi)| \lesssim |\xi|^{-|\alpha|}$ for $0 \leq |\alpha| \leq d+2$. In this case, we will show that $|\nabla m^\vee(x)| \lesssim |x|^{-(d+1)}$ uniformly for $|x| \neq 0$. This yields (c).

We have

$$\||x^\alpha|\nabla m^\vee(x)\|_{L_x^\infty} \lesssim \underbrace{\left\|D_\xi^\alpha[\xi m(\xi)]\right\|_{L_\xi^1}}_{O(|\xi|^{1-|\alpha|})},$$

But this is not integrable! However, we can integrate it on dyadic annuli. Write

$$m(\xi) = \sum_{N\in 2^{\mathbb{Z}}} m_N(\xi), \qquad m_N(\xi) = m(\xi)\psi_N(\xi).$$

Then the chain rule gives

$$D_\xi^\alpha[\xi m_N(\xi)] = \sum_{\alpha_1+\alpha_2=\alpha} D_\xi^{\alpha_1}[\xi m(\xi)] D_\xi^{\alpha_2}[\psi_N(\xi)],$$

so

$$|D_\xi^\alpha[\xi m_N(\xi)]| \lesssim_\alpha \sum_{\alpha_1+\alpha_2=\alpha} |\xi|^{1-|\alpha_1|} N^{-|\alpha_2|} |D_\xi^{\alpha_2}\psi|(\xi/N).$$

Then

$$\begin{aligned}
\| |x^\alpha | \nabla m_N^\vee(x)\|_{L_x^\infty} &\lesssim \|D_\xi^\alpha [\xi m_n(\xi)]\|_{L_\xi^1} \\
&\lesssim_\alpha \sum_{\alpha_1+\alpha_2=\alpha} \int_{|\xi|\sim N} |\xi|^{1-|\alpha_1|} N^{-|\alpha_2|}\, d\xi \\
&\lesssim_\alpha N^{1-|\alpha|+d}.
\end{aligned}$$

So we get

$$|\nabla m_N^\vee(x)| \lesssim \min\{N^{d+1}, (N|x|^{d+2})^{-1}\}.$$

By the triangle inequality,

$$\begin{aligned}
|\nabla m^\vee(x)| &\le \sum_{N\in 2^{\mathbb{Z}}} |\nabla m_N^\vee(x)| \\
&\lesssim \sum_{N\le |x|^{-1}} N^{d+1} + \sum_{N>|x|^{-1}} \frac{1}{N|x|^{d+2}} \\
&\lesssim |x|^{-(d+1)},
\end{aligned}$$

uniform in $|x| \neq 0$.

Now let's prove condition (c) assuming this hypothesis holds for only $0 \le |\alpha| \le \lceil \frac{d+1}{2} \rceil$. Look at

$$\int_{|x|\ge 2|y|} |m^\vee(x+y) - m^\vee(x)|\, dx = \sum_{N\in 2^{\mathbb{Z}}} \int_{|x|\ge 2|y|} |m_N^\vee(x+y) - m_N^\vee(x)|\, dx$$

If we have $\widehat{f}_N(\xi) = \widehat{f}(\xi)\psi_(\xi)$, then $f_N(x) = (f * N^d\psi(N\,\cdot))(x) = \int f(x-y)N^d\psi^\vee(Ny)\,dy$, so $|f_N(x)| \lesssim \int |f(x-y)|N^d \frac{1}{\langle Ny\rangle^m}\,dy$.

$$\begin{aligned}
&\lesssim \sum_{N\le |y|^{-1}} \int_{|x|\ge 2|y|} |m_N^\vee(x+y) - m_N^\vee(x)|\, dx \\
&\quad + 2\sum_{N>|y|^{-1}} \int_{|x|\ge |y|} |M_n^\vee(x)|\, dx
\end{aligned}$$

Using the fundamental theorem of calculus,

$$\begin{aligned}
&\lesssim \sum_{N\le |y|^{-1}} \int_{|x|\ge 2|y|} |y| \cdot \int_0^1 |\nabla m_N^\vee(x+\theta y)|\, d\theta\, dx \\
&\quad + 2\sum_{N>|y|^{-1}} \int_{|x|\ge |y|} |m_N^\vee(x)|\, dx.
\end{aligned}$$

We will complete the proof next time. □